Competing processes on hyperbolic non-amenable graphs

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(joint work(s) with A.Stauffer)

Università degli Studi Roma 3

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Model

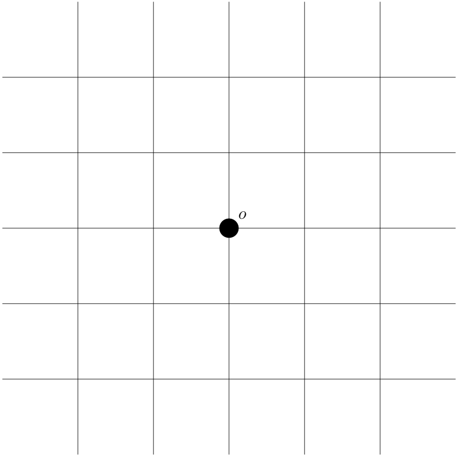
We deal with 2 First-passage percolation processes:

$$\mathbf{FPP}_1 \quad \text{and} \quad \mathbf{FPP}_\lambda$$

spreading on a graph G.
Choose 2 parameters: $\boldsymbol{\lambda} > 0$ and $\boldsymbol{\mu} \in (0, 1)$.

- At the origin o place a **black** particle;
- At every $x \in V(G) \setminus \{o\}$ place a **red** particle (call it **SEED**) with probability $\boldsymbol{\mu}$ independently for all vertices.

Model: Example on $\mathbb{Z}^2$



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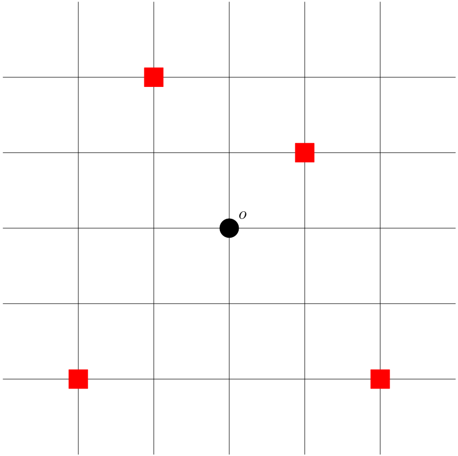

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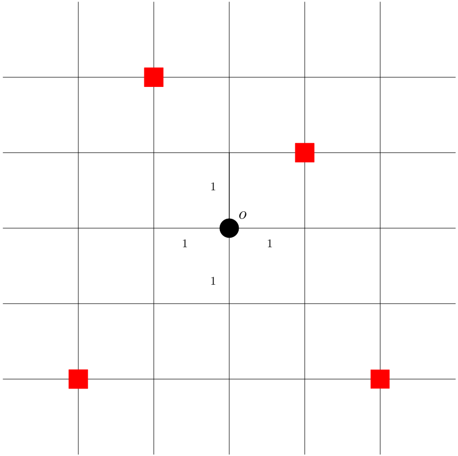

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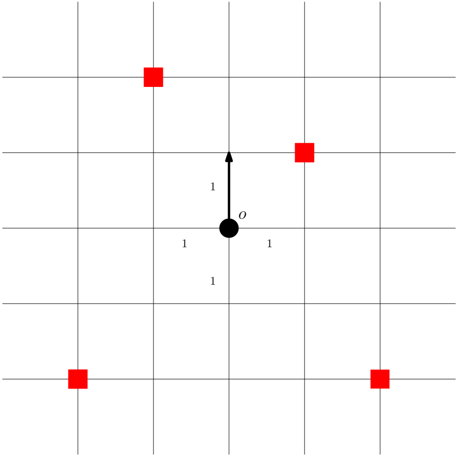

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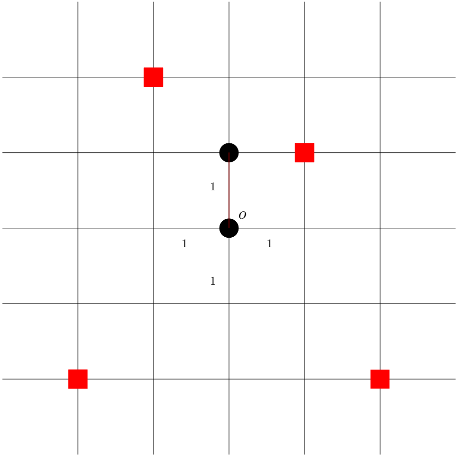

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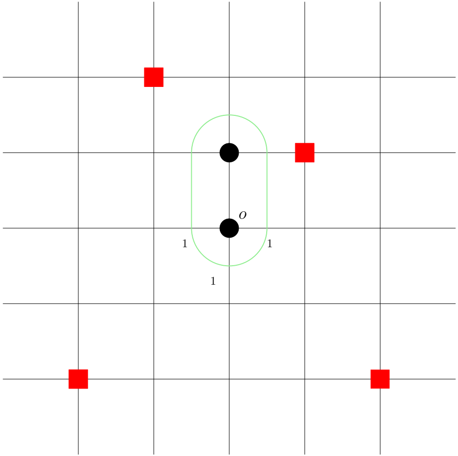

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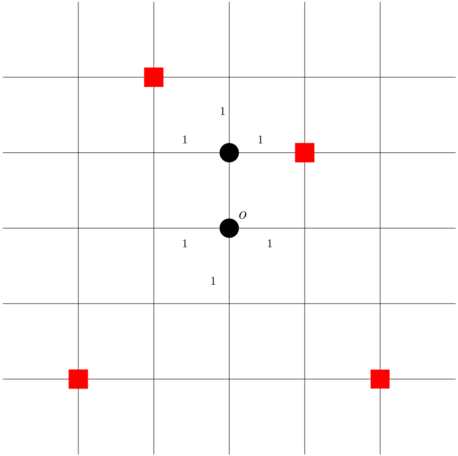

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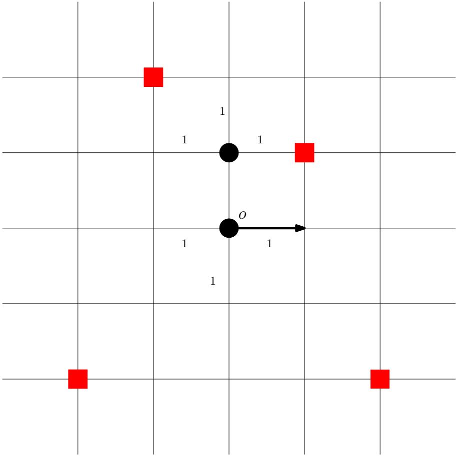

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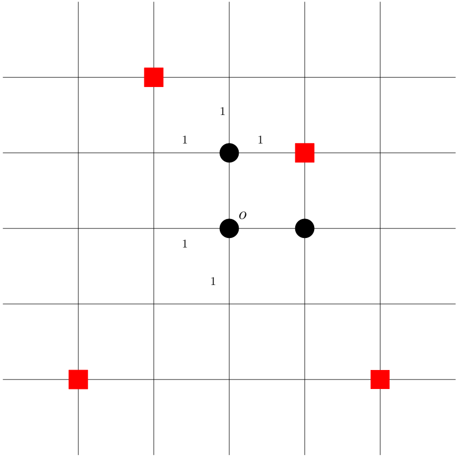

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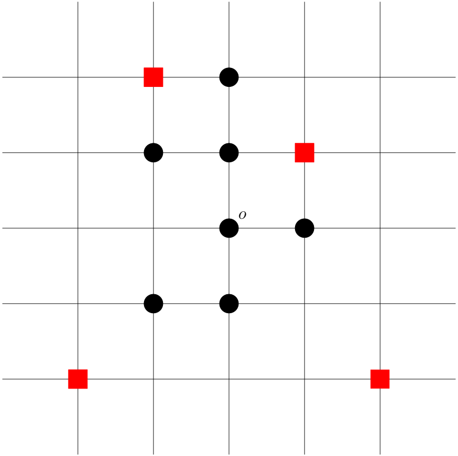

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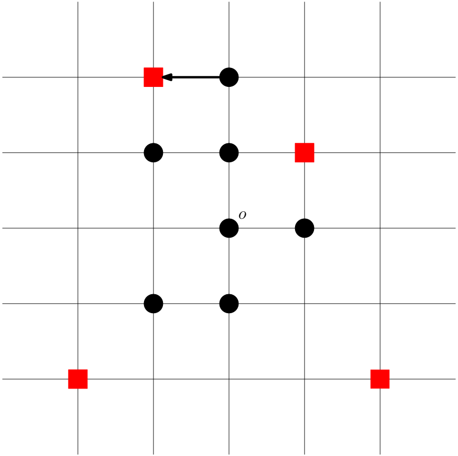

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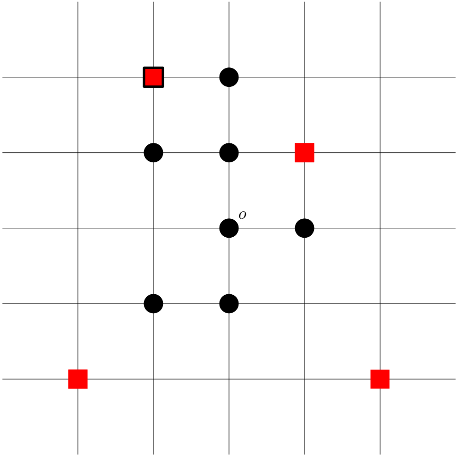

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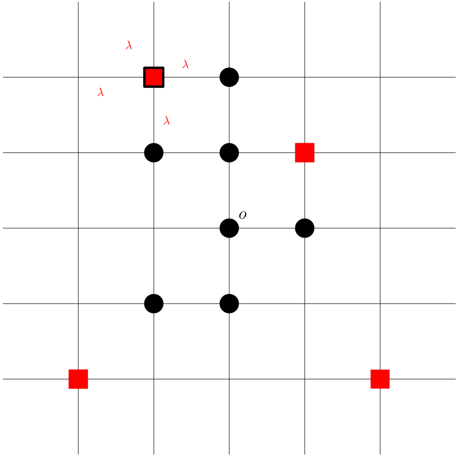

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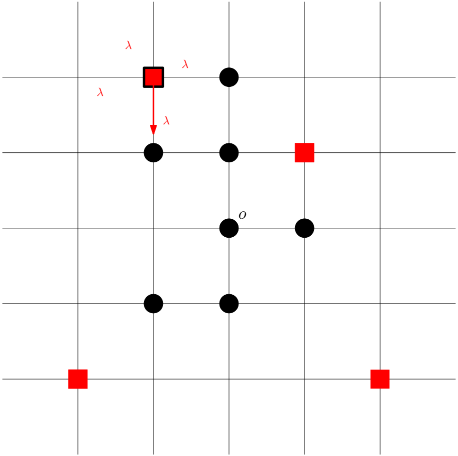

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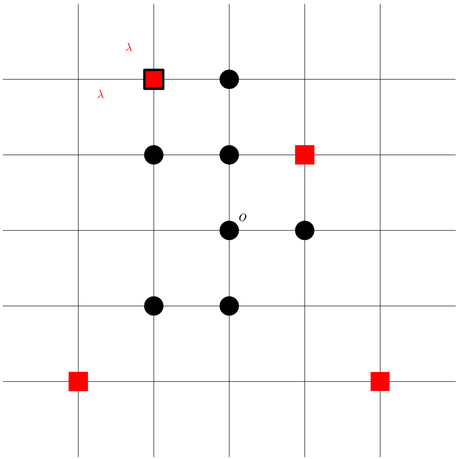

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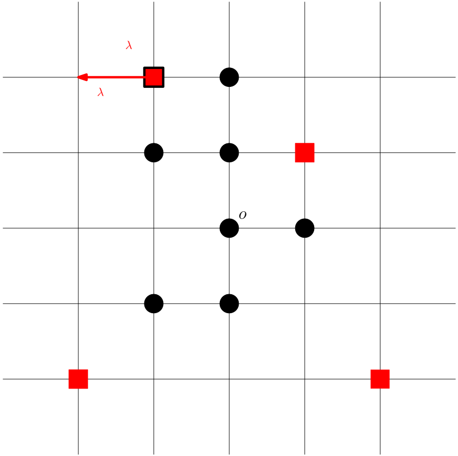

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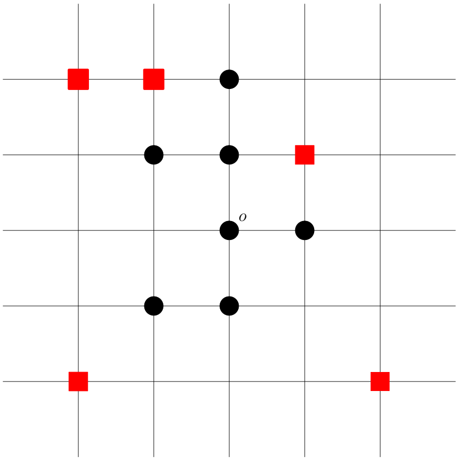

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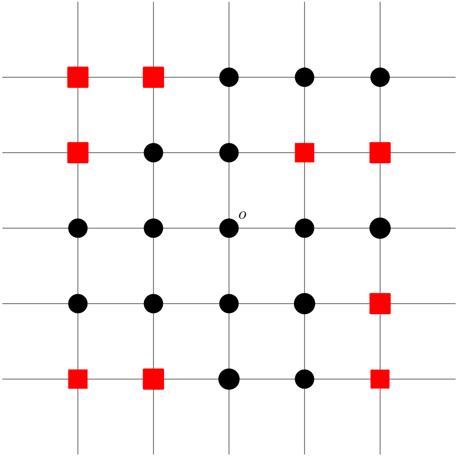

Dynamics

$\mathbf{FPP}_1$: starts at o and has passage times $\sim$ Exp(1).

Inductively, place an exponential clock of rate 1 on all edges neighboring the current **black cluster** and look at which one rings first, then "grow" the black cluster in that direction.

In the meanwhile, **seeds are inactive (sleeping)**.

$\mathbf{FPP}_\lambda$: when a seed is *activated*, it starts spreading FPP at rate $\boldsymbol{\lambda} > 0$.

NOTE: Occupied vertices stay so for ever.

Model/Questions

We have these two processes (think of infections) that spread at different rates (**black** at rate **1** and **red** at rate $\boldsymbol{\lambda} > 0$) and are competing for space.

Some questions about the model:

- **Survival**: When either process occupies an INFINITE CONNECTED region of the graph
- **Coexistence**: When both processes survive simultaneously
- **Monotonicity**: Probability of $\mathbf{FPP}_1$ surviving is/isn't monotone in μ

Related works

Model introduced by Sidoravicius and Stauffer (*Invent. Math.*) as

First-Passage Percolation in Hostile Environment

to understand MDLA on $\mathbb{Z}^d$, $d \geq 2$. (Coupling MDLA with FPPHE)

Theorem [Sidoravicius and Stauffer, 2019] On $\mathbb{Z}^d$ for $d \geq 2$, for all $\boldsymbol{\lambda} \in (0,1)$ there is $\mu_0 = \mu_0(d, \boldsymbol{\lambda}) > 0$ such that when $0 < \boldsymbol{\mu} < \mu_0$:

$$\mathbb{P}_{\boldsymbol{\mu}}\left[\mathbf{FPP}_1 \text{ survives and } \mathbf{FPP}_\lambda \text{ does not}\right] > 0.$$

Related works

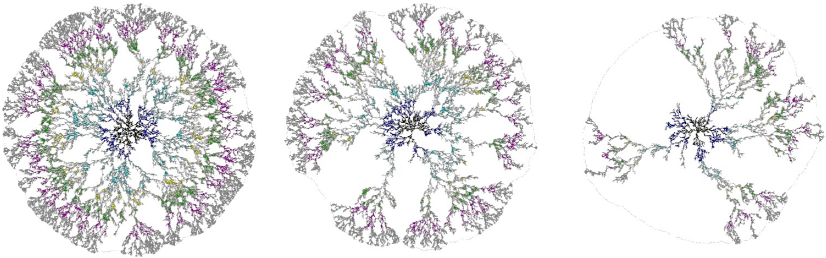

Figure: FPPHE on $\mathbb{Z}^2$ with $\lambda = 0.7$ and $\mu = 0.027, 0.029, 0.030$ respectively. Colors $\Rightarrow$ different epochs of the growth of $\mathbf{FPP}_1$. The whole white region within the thin boundary is occupied by activated $\mathbf{FPP}_\lambda$.

Related works

Theorem [Finn and Stauffer, 2021+] On $\mathbb{Z}^d$ for $d \geq 3$, there are values $0 < \mu_0 < \mu_1 < 1$ such that for $\mu_0 < \boldsymbol{\mu} < \mu_1$ and $\boldsymbol{\lambda}$ small enough, then

$$\mathbb{P}_{\boldsymbol{\mu}}\left[\mathbf{FPP}_1 \text{ and } \mathbf{FPP}_\lambda \text{ coexist}\right] > 0.$$

Not clear what happens for $d = 2$.

Our setting: hyperbolic and non-amenable graphs

Hyperbolic graphs

A graph is δ hyperbolic (for some $\delta \geq 0$) if for any triplet of vertices $\{x, y, z\}$:

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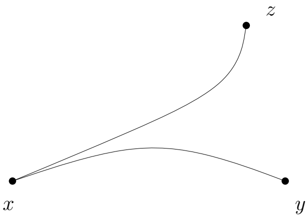

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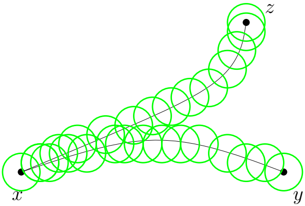

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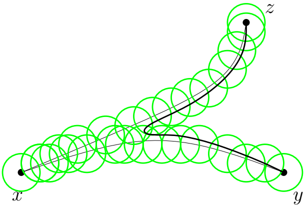


Our setting: hyperbolic and non-amenable graphs

Non-amenable graphs

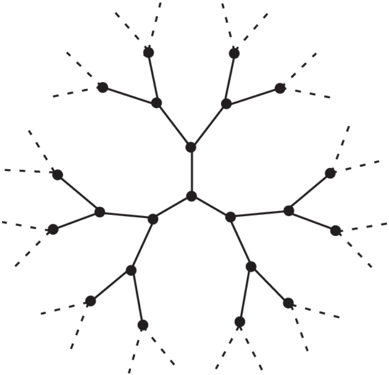
For all finite sets of vertices A, let

$$\partial A := \{x \in A \ : \ \exists\, y \notin A, \{x, y\} \in E(G)\}.$$

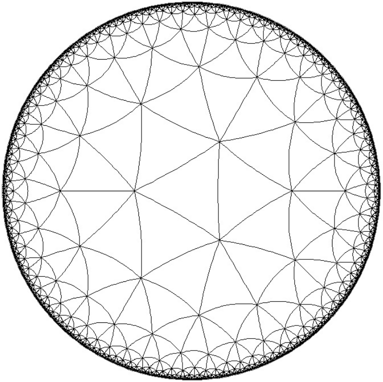
G is non-amenable if there is a constant $\mathbf{c} > 0$ such that

$$\inf_{|A|<\infty} \frac{|\partial A|}{|A|} \geq \mathbf{c}.$$

Hyperbolic non-amenable graphs: Two Examples



Infinite binary tree



Hyperbolic tessellation

Intermezzo: Why do we care about such graphs?

FPP can be seen as a model for **spread of infection**, or the **spread of a false rumor within a network**. Think of the following:

Misinformation (as $\mathbf{FPP}_1$) starts from an individual in the network.

When *detected* by the detecting stations (*seeds*), they try to stop it.

Proven: there are models for real-world networks with intrinsic *hyperbolic* and (local) *non-amenable* properties.

Monotonicity in μ?

It turns out that (percolation arguments)

(for all λ) if μ very close to 1 $\Rightarrow$ **FPP**$_1$ a.s. doesn't survive,

whereas (at least on $\mathbb{Z}^d$ as seen before)

if λ and μ small enough $\Rightarrow$ **FPP**$_1$ survives w.p.p.

Moreover, seeds get in the way of **FPP**$_1$ because they can "interrupt" **black** paths.

Monotonicity in μ? – Natural conjecture

Thus it is natural to conjecture that for **small** λ

$$\mathbb{P}_{\mu}(\mathbf{FPP}_1 \text{ survives}) \text{ is monotone in } \mu.$$

Monotonicity in μ? – Natural conjecture

Thus it is natural to conjecture that for **small** λ

$$\mathbb{P}_{\mu}(\mathbf{FPP}_1 \text{ survives}) \text{ is monotone in } \mu.$$

FALSE!

Results on Monotonicity in μ

Theorem [C. and Stauffer, 2021+] There is an infinite (hyperbolic and non-amenable) graph G s.t. we can find two values $0 < \mu_1 < \mu_2 < 1$:

$$\mathbb{P}_{\mu_1}(\mathbf{FPP}_1 \text{ survives}) = 0,$$

and

$$\mathbb{P}_{\mu_2}(\mathbf{FPP}_1 \text{ survives}) > 0.$$

$\Rightarrow$ (Coming soon on Arxiv, stay tuned!) $\Leftarrow$

This means that adding seeds might actually *help* $\mathbf{FPP}_1$!

Lots of troubles in applying methods e.g. multi-scale renormalization!

Survival and coexistence on hyperbolic graphs

Theorem [C. and Stauffer, 2021]

(i) Let G be infinite, **hyperbolic**, **non-amenable**, locally finite, then for all $\boldsymbol{\lambda} > 0$ there is $\mu_0 = \mu_0(G, \boldsymbol{\lambda}) > 0$ such that when $0 < \boldsymbol{\mu} < \mu_0$:

$$\mathbb{P}_{\boldsymbol{\mu}}[\mathbf{FPP}_1 \text{ survives indefinitely}] > 0.$$

(ii) Let G be infinite, **hyperbolic**, **vertex-transitive**, locally finite, then for all $\boldsymbol{\mu} \in (0, 1)$, for every $\boldsymbol{\lambda} > 0$ we have

$$\mathbb{P}_{\boldsymbol{\mu}}[\mathbf{FPP}_\lambda \text{ survives indefinitely}] = 1.$$

Survival and coexistence on hyperbolic graphs

Corollary [C. and Stauffer, 2021] Let G be infinite, **hyperbolic**, **non-amenable**, **vertex-transitive**, locally finite, then:
for all $\boldsymbol{\lambda} > 0$ there is $\mu_0 = \mu_0(G, \boldsymbol{\lambda}) > 0$ such that when $0 < \boldsymbol{\mu} < \mu_0$:

$$\mathbb{P}_{\boldsymbol{\mu}}[\textbf{FPP}_1 \text{ and } \textbf{FPP}_\lambda \text{ coexist}] > 0.$$

Proof.

Immediate consequence of the above Theorem. □

Survival and coexistence on hyperbolic graphs

On the "board":

Sketch of proof of **Theorem [C. and Stauffer, 2021]**

(ii) Let G be infinite, **hyperbolic**, **vertex-transitive**, locally finite, then for all $\mu \in (0,1)$, for every $\lambda > 0$ we have

$$\mathbb{P}_{\mu}[\textbf{FPP}_{\lambda} \text{ survives indefinitely}] = 1.$$

Practically

If you try to spread a **false rumor on** $\mathbb{Z}^d$:

Few detectors + slow spread of the truth $\Rightarrow$ false rumor can overcome all efforts of the detectors!

If you try to spread a **false rumor on a hyperbolic & non-amenable graph**:

The efforts of the detectors will always pay off!

Thank you for your attention!